Defence Technology 13 (2017) 177–187

Contents lists available at ScienceDirect

Defence Technology

journal homepage: www.elsevier.com/locate/dt

Improved theory of projectile trajectory reference heights as characteristics of meteo-ballistic sensitivity functions

Vladimir Cech [a, c, *], Jiri Jevicky [b, c]

[a] Department of Weapons and Ammunition, University of Defence, Kounicova 65, 662 10 Brno, Czech Republic
[b] Department of Mathematics and Physics, University of Defence, Kounicova 65, Brno 662 10, Czech Republic
[c] Oprox, Inc., Kulkova 8, Brno 615 00, Czech Republic

ARTICLE INFO

Article history:
Received 29 January 2017
Received in revised form
24 March 2017
Accepted 10 April 2017
Available online 12 April 2017

Keywords:
Exterior ballistic
Sensitivity analysis
Non-standard projectile trajectory
Green's functions
Weighting factor functions (curves)
Reference height of projectile trajectory
METEO-11 ("meteo-average")

ABSTRACT

Projectile trajectories calculated under non-standard conditions are considered to be perturbed. The tools utilized for the analysis of perturbed trajectories are weighting factor functions (WFFs) which are a special kind of sensitivity functions. WFFs are used for calculation of meteo ballistic elements μ_B (ballistic wind w_B, virtual temperature τ_B, pressure p_B, density ρ_B, speed of sound a) as well. An effect of weapon system parameters can be incorporated into calculations through the reference height of trajectory - RHT. RHT are also calculated from WFFs. Methods based on RHT are far more effective than traditional methods that use weighting factors q.

We have found that the existing theory of RHT has several shortcomings due to we created an improved theory of generalized RHT which represent a special sensitivity parameters of dynamical systems. Using this theory will improve methods for designing firing tables, fire control systems algorithms, and meteo message generation algorithms.

1. Introduction

This contribution follows up on our earlier publications [1–5], but a detailed analysis of problems under consideration is presented in studies [6–8]. For the sake of understanding of contents of this article, it is, at least, needful to peruse problems of weighting (factor) functions (curves) WFFs presented in Ref. [4]. The traditional theory of the reference height of a trajectory RHT is elaborated in the article [1] predominatingly under utilization of the textbook [10].

We apologize for erroneously referred to signs in the article [4] in the relations (33–35). In the present article they correspond to the relations (2–4), in which the signs are placed correctly. Moreover, in Ref. [4] in the relation (45): before the square bracket has to be the right number −1.

1.1. Motivation

We continue in research with the same theme and therefore our motivation can not change [1–5]: It follows from the analysis of artillery fire errors, e.g. Refs. [9,10], that approximately two-thirds of the inaccuracy of indirect artillery fire is caused by inaccuracies in the determination of meteo parameters included in the error budget model [9]. Consequently, it is always important to pay close attention to the problems of including the actual meteo parameters in ballistic calculations [10]. The following meteo parameters μ are primarily utilized: Wind vector $\boldsymbol{w}$, air pressure p, virtual temperature τ, density ρ, and speed of sound a [10–13].

This paper deals only with problems relating to unguided projectiles without propulsion system for the sake of lucidity of the solved problems.

List of notation	
μ	met parameter (element)
$\mu(y)$	real or measured magnitude of met parameter μ in height y
$r(\mu)$	weighting factor function (curve, WFF)
Q_P, Q_{CP}	effect function
$\mu_{STD}(h)$	met parameter standard course with the height h
$\Delta\mu(y)$	absolute deviation of met element μ in height y
$\delta\mu(y)$	relative deviation of met element μ in height y
$\Delta\mu_B$	absolute ballistic deviation of ballistic element μ_B
$\delta\mu_B$	relative ballistic deviation of ballistic element μ_B

* Corresponding author. Oprox, Inc., Kulkova 8, Brno 615 00, Czech Republic.
E-mail addresses: cech-vladimir@volny.cz (V. Cech), jiri.jevicky@centrum.cz (J. Jevicky).

Peer review under responsibility of China Ordnance Society.

http://dx.doi.org/10.1016/j.dt.2017.04.001

1.2. References and terminology

The reference height RHT is only in use by the Soviet methodology, which is joined with utilization of the meteo message METEO-11 ("Meteo-average"), and therefore references to this area are totally missing in publications which were published outside the former Soviet bloc. It is a problem that only the textbook [10] can be considered as quality. The next publications [14–20] give only a slight information which can serve to the understanding of basic properties of the reference height RHT and methods of its utilization. Only the article [20] deals with a new method for computation of the reference height RHT.

The new methodology was introduced (see Ref. [20]) by the instruction: "Наставление артиллерии советской армии, Артиллерийская метеорологическая служба, Москва 1956" ("Artillery Instruction of the Soviet Army, Artillery Meteorological Service, Moscow 1956"). Referred material is for us yet unavailable. The methodology was gradually introduced even in the 1960s and 1970s in the Soviet satellites. In the Soviet literature there is used the original designation for the RHT "высота входа в бюллетень "метеосредний"" ("input height into bulletin "meteo-average"") or the short one "условная высота" ("conventional height"). The first designation is too long and the second one does not correspond to the physical nature of this quantity, therefore we decided to use other designations. The author of the articles [19,20] uses the sign "conditional height", which is also not too appropriate.

1.3. The main objectives of the contribution

We have shown in articles [1–3] that the Soviet methodology for the use of the reference height RHT and the meteo message METEO-11 ("meteo-average") has a greater potential for increasing the accuracy of the calculation of the corrections on the influence of meteo parameters than the traditional methodology [21–26], which was introduced already in the 1910s and 1920s and which uses a system of weights $q(\mu)$ (weighting factors – WFs) derived from the relevant WFFs. The formation and use of the meteo message METBKQ [27] is based on this methodology and also the corresponding creation and use of the tabular firing tables – TFT in practice [28–33]. The Soviet methodology used until 1956 was virtually identical, too.

If the METEO-11 is in use, then the weighting factors $q(\mu)$ are proportional only to the relative height of the individual zones – layers. The effect of ballistic parameters of weapons and ammunition is then included in the calculations just by using the reference height of projectile trajectory – RHT Y_R, which is a multiple of the height of trajectory (HT, $Y = Y_S$), thus $Y_R = K_R \cdot Y_S$, $K_R =$ cc. 0.8 to 1.4. The values of RHT are together with HT given in the tabular firing tables for the relevant projectile and the charge depending on the range of fire. The reference heights RHT are calculated also from the relevant WFF.

In the meteo message METBKQ, the ballistics elements μ_B are sought for the height of trajectory HT Y_S corresponding to the given range X, whereas in the METEO-11, the ballistics elements μ_B are sought for the reference height RHT Y_R corresponding also to the given range X.

We have shown in the article [4] that from the view of the modern theory of sensitivity analysis of dynamical systems, the WFFs are normed sensitivity functions, which provide concentrated information about the sensitivity of changes of projectiles trajectories on the perturbations of the relevant meteo parameters μ.

The weighting factors and the reference heights RHT are derived from the WFFs and therefore they also pertain to the sensitivity characteristics of the meteo-ballistic system.

In the available literature, there exists a semi-empirical definition of the reference height RHT [10,19]. We analyzed this problem in Refs. [1–3,6]. An exact definition has not yet been published.

The main contribution of the article is an exact derivation of the definition relationship for the reference height RHT – the relation (40). This relation defines the generalized RHT [7,8], whose special case is the traditional RHT.

In practice, three perturbation models are used for the calculation of the WFFs [26,4]. The soviet methodology uses a model formulated by Frenchman P. Langevin [4,10,26]. In this case, the WFFs are without the norm effect or with a weak norm effect, therefore, the authors of the original theory of the reference height of trajectory (RHT) at all did not deal with problems of the norm effect.

Our goal was to modify the theory of RHT so as to be applicable even for perturbation model for the calculation of the WFFs used initially in the USA and now also in NATO [23,25,26]. It is typical for this model that some WFFs are with the strong and exact norm effect – paragraph 1.4.2. In this case, division by zero or by very small numbers occur in the calculations.

The practical consequence is that the calculation of respectively corrections and gradients for the numerical iteration in a neighbourhood of a strong norm effect and the exact one, either completely fails, or is very inaccurate.

Our contribution is that we can control the whole process of numerical calculations so that the needed accuracy should be achieved also in areas with strong and exact effects. Whereas, it is given "only" by the fact, that as a parameter, in all formulas, there is performing consistently the value $r(1)$ as a measure of the intensity of the norm effect.

Subsequently, we analyze three ways of numerical calculation of the generalized reference height RHT:

- our original procedure based on the definition relationship (40) – the paragraphs 2.2 and 2.3,
- the original procedure modified by us [6–8,10] – the paragraph 3.3. and relationship (65),
- the procedure proposed by Petrovic in Ref. [20], which we also generalized [8] – the paragraph 4.1. and relationship (72) and the paragraph 4.2. and relationship (78).

The published results allow generalizing of the original soviet methodology and widely putting into practice.

1.4. Weighting functions - basic information

In this paragraph, we briefly present findings from our article [4], which are necessary for the understanding of the subsequently presented theory of the reference heights RHT.

1.4.1. Green's functions of the projectile trajectory sensitivity models

The best way to analyze the characteristics of the projectiles trajectories under nonstandard conditions is the build of any of the explicit sensitivity models of projectile trajectory [21,23,25,26]. The talk is then about the (differential) sensitivity analysis of dynamical system (projectile trajectory) or about the parameter sensitivity analysis or about the sensitivity of a system to parameter variations.

The perturbation theory is often used to the build of these models [34,35]. These are the linearized models represented by systems of linear differential equations with variable coefficients.

Relations between generalized inputs (control input variables, disturbance input variables and variable parameters of the system) $z_m(t)$, $m = 1, 2, \ldots$ on one side and output variables y_l, $l = 1, 2, \ldots$ on the other hand, are given traditionally by transfer functions and Green's functions $g_{m,l}(t{-}t_P)$. There is also a generalized theory of

Green's functions for some groups of non-linear systems.

In our case, we take into consideration only the following generalized inputs: the wind vector $\boldsymbol{w} = (w_x, w_y, w_z)$ as a disturbance input variable and next variable meteo parameters (the virtual temperature τ, the air pressure p, the air density ρ and the speed of sound a). All we denote, as we have already noted, by the common symbol μ. In the case of other parameters of the model of the projectile trajectory, it progresses as well.

The most important output variables are the coordinates of the partially perturbed projectile trajectory (x, y, z) and corresponding time of flight t.

Green's functions are also denoted as weight or weighting functions or influence functions or impulse response functions. In the case of sensitivity models of a dynamical system, the Green's functions represent special sensitivity functions of two parameters (t, t_P):

t is the moment to which the system response will be calculated. In our case, it is the moment of t_{PI} in which the standard projectile trajectory passes through chosen point of impact/burst with coordinates $(t, x, y, z)_{PI}$. According to its position on the projectile trajectory, we distinguish four types of the projectile trajectories [4].

t_P is the moment in which to impress $z_m(t - t_P) = z_m(t) + \Delta z_{m0}\cdot\delta(t - t_P)$, where $z_m(t)$ is unperturbed quantity, Δz_{m0} is the amplitude of excitation and $\delta(t - t_P)$ the unit impulse (the Dirac delta function). We distinguish three types of trajectories of the projectile:

- partially perturbed trajectory, if $t_P \in (0, t_{PI})$ and its special variants:
- (full) perturbed trajectory), if $t_P = 0$, i.e. the impulse impresses at the beginning of the trajectory and
- the standard (unperturbed) trajectory, if $t_P \geq t_{PI}$.

In the exterior ballistics there are used (traditionally from the 1910s) the effect functions $R_{m,l}$ as the response to the excitation $z_m(t - t_P) = z_m(t) + \Delta z_{m0}\cdot H(t - t_P)$, where $z_m(t)$ is unperturbed quantity, Δz_{m0} is the amplitude of the excitation, and $H(t - t_P)$ is the Heaviside step function.

The normed effect functions are denoted as the weight or weighting functions, it is therefore a different terminology due to the theory of dynamical systems. To differentiate them from the Green's functions, we will refer to weighting (factor) functions (curves) WFFs $r_{m,l}$ [10,14–21,23,25,26].

Due to the properties of the Dirac delta function and the Heaviside step function, the following relationship applies [4,7]

$$g_{ml}(t - t_P) = -\frac{dR_{ml}}{dt_P} = -\left(\sigma_{ml}\cdot\frac{N_{ml}}{\Delta z_{m0}}\right)\cdot\frac{dr_{ml}}{dt_P}, \quad (1)$$

where

N_{ml} is the relevant norm – see the paragraph 1.4.2. and
$\sigma_{ml} = +1$ or -1 is the contractual sign – see the paragraph 1.4.2.

It follows from this relation that not only the Green's functions are sensitivity functions, but also the effect functions $R_{m,l}$ and the weighting functions WFFs $r_{m,l}$ are sensitivity functions, too.

In practice [4,7], the functions $Q_{Pm,l} = \Delta z_{m0}\cdot R_{m,l}$ are taking rather than the functions $R_{m,l}$, and they are also referred to as effect functions. We will use mentioned functions hereafter and we will denote them $Q_P(\mu, t_P) = Q_P(\mu) = Q_P(t_P) = Q_P$ [4,7,10]. Under this designation, we understand especially special shapes of the perturbation of the output quantities Δy_l: $\Delta X(\mu, t_P)$, $\Delta Y(\mu, t_P)$, $\Delta Z(\mu, t_P)$ a $\Delta t(\mu, t_P)$. The values are relative to the moment t_{PI} in which the standard trajectory passes through the point of impact/burst [4,7]. It will be assigned to the functions $Q_P(\mu, t_P)$ the norm $N_{m,l}$ and the sign σ_{ml}, we will denote them N_Q and σ_Q henceforth.

The effect functions $Q_P(\mu, t_P)$ and the corresponding weighting functions WFFs $r(\mu, t_P)$ are functions of two times (t_{PI}, t_P) and the corresponding meteo parameter $\mu(t_P)$, which is subject to the perturbation. In the basic variant of analyses, we assume that only one selected meteo-parameter is subject to the perturbation.

The total perturbation $Q_{PS}(\Delta\mu)$ of the parameters of the point of impact/burst, e.g. the range $\Delta X_P(\Delta\mu)$ – if the course of the absolute deviations $\Delta\mu(t)$ from the standard values is known (measured), $\Delta\mu(t) = \mu(t) - \mu_{STD}(t)$ – is given by the convolutory integral [4,7]

$$\Delta X_P \sim Q_{PS}(\Delta\mu) \cong -\int_0^{t_{PI}} \Delta\mu(t_P)\cdot\frac{dQ_{PA}(\mu, t_P)}{dt_P}\cdot dt_P$$
$$= (\sigma_Q\cdot N_Q)\cdot\left(\frac{\Delta\mu_B}{\Delta\mu_{B0}}\right), \quad (2)$$

and

$$\Delta\mu_B \cong -\int_0^{t_{PI}} \Delta\mu(t_P)\cdot\frac{dr_A(\mu, t_P)}{dt_P}\cdot dt_P, \quad (3)$$

where

$\mu_{STD}(t)$ is the standard course of the meteo parameter μ,
$\Delta\mu_{B0}$ = const is a contractual value [4,7],
$\Delta\mu_B$ is the absolute ballistic deviation/perturbation of the ballistic meteo parameter μ (e.g. ballistic range/cross wind $(w_x, w_z)_B$, absolute ballistic virtual temperature deviation/perturbation τ_B, etc.).

Analogous relations to (2) and (3) also applies for the known (measured) relative deviations $\delta\mu(t)$ $(\delta\mu(t) = \Delta\mu(t)/\mu_{STD}(t))$ [4,7]. In that case, the effect functions and WFFs will have index R (relative deviation $\delta\mu$) instead of the index A (absolute deviation (perturbation) $\Delta\mu$ of the meteo parameter μ), i.e. Q_{PR}, r_R.

All the following considerations are identical for respectively the absolute and relative deviations (perturbations) of the meteo parameter μ, therefore, we will do the derivation only for the variant "absolute deviations (perturbations) $\Delta\mu$ of meteo parameter μ", while the index A will be omitted.

Measured deviations are evaluated for the requirements of the flat-fire depending on the topographic range, i.e., on the coordinate x, and so $(\Delta\mu(x), \delta\mu(x))$ is used [4,7]. As a consequence, the equations (2) and (3) must be modified.

We will use the function $t_P = F(x)$, which is valid for the standard trajectory; then it will be $Q_P(\mu, x)$ and $r(\mu, x)$. Let us remind the reader that $dx = v_x\cdot dt_P$, thus the equation (3) will have the form

$$\Delta\mu_B \cong -\int_0^{x_{PI}} \Delta\mu(x)\cdot\frac{dr_A(\mu, x)}{dx}\cdot dx \quad (4)$$

We choose $(\sigma_Q\cdot N_Q) = Q_P(\mu, x)$ for $x = 0$ in all cases $(N_Q = \mathrm{abs}(Q_P(0)))$, $\sigma_Q = \mathrm{sign}(Q_P(0)))$.

The WFFs for the range wind $r_{wx}(x)$ and for the cross wind $r_{wz}(x)$ are important only for the flat-fire from a practical point of view.

For shooting at common trajectories, measured deviations $(\Delta\mu, \delta\mu)$ are evaluated depending on coordinate y of the projectile trajectory, thus $(\Delta\mu(y), \delta\mu(y))$ is used [4,7]. Therefore it is necessary to modify the relations (2 and 3) again.

It was newly created a generalized theory of the generalized effect functions $Q_{CP}(\mu, t_P)$ and the generalized WFFs $r_C(\mu, t_P)$ [4,7].

From the generalized effect functions $Q_{CP}(\mu, t_P)$ are successively calculated generalized Garnier's effect functions $Q_{CG}(\mu, y)$ and generalized Bliss' effect functions $Q_{CB}(\mu, y)$, which are already functions of the vertical coordinate y [4,7]. This conversion relationship holds

$$Q_{CB}(y) = Q_{CB}(\mu, y) = Q_{CG}(\mu, y_{min}) - Q_{CG}(\mu, y) \tag{5}$$

The value $Q_{CG}(\mu, y_0)$ represents the cumulative effect of all perturbations in heights $y \geq y_0$ and the value $Q_{CB}(\mu, y_0)$ represents the cumulative effect of all perturbations in heights $y \leq y_0$.

Generalized weighting factor functions WFFs are calculated by norming from generalized effect functions (curves).

For generalized Garnier's weighting factor functions WFFs it holds that [4,7]

$$r_{CG}(\mu, y) = \frac{Q_{CG}(\mu, y)}{(\sigma_Q \cdot N_Q)}, \tag{6}$$

For generalized Bliss' weighting factor functions WFFs it holds that [4,7]

$$r_{CB}(\mu, y) = \frac{Q_{CB}(\mu, y)}{(\sigma_Q \cdot N_Q)} = r_{CG}(\mu, y_{min}) - r_{CG}(\mu, y), \tag{7}$$

where

$$r_{CG}(\mu, y_{min}) = \frac{Q_{CG}(\mu, y_{min})}{(\sigma_Q \cdot N_Q)} = r_{CB}(\mu, y_{max}),$$

where

$y_{min} = \min(y(t))$ and $y_{max} = \max(y(t))$ for $t \in \langle 0, t_{PI} \rangle$.
$Q_{CG}(\mu, y_{min}) = Q_{CP}(\mu, y_{min}) = Q_{CP}(\mu, t_P) = Q_{CP}(t_P)$ for $t_P = 0$ and
$r_{CG}(y_{max}) = r_{CB}(y_{min}) = 0$.

Furthermore the relationship holds

$$\frac{r_{CB}(\mu, y)}{dy} = -\frac{r_{CG}(\mu, y)}{dy} \tag{8}$$

According to NATO and Soviet methodologies using Bliss' WFFs $r_B(\mu, y)$ are presupposed, so we will limit our following analysis only to generalized Bliss' WFFs $r_{CB}(\mu, y)$. If interchange is not possible, we will no longer mention index "CB" in description of WFFs.

However, the generalized Garnier's WFFs are preferable for analyses and graphical display of the so-called "norm effect" (see the paragraph 1.4.2).

Analogically for relations (2 and 3), ($dy = v_y(t_P) \cdot t_P$) will apply [4,7]

$$\Delta\mu_B \cong \int_{y_{min}}^{y_{max}} \Delta\mu(y) \cdot \frac{dr_{A,CB}(\mu, y)}{dy} \cdot dy \tag{9}$$

It is convenient to introduce the following normalization of the vertical coordinate y

$$\eta = \frac{y - y_{min}}{y_{max} - y_{min}} = \frac{h - h_{min}}{h_{max} - h_{min}} = \frac{\Delta h}{\Delta h_m} \tag{10}$$

where

$h = h_G + y$ is the altitude corresponding to the vertical coordinate of the y,
h_G is the altitude of the horizontal plane (x, z), $y = 0$ of the ballistic system,
$\Delta h = h - h_{min} = y - y_{min}$ and
$\Delta h_m = h_{max} - h_{min} = y_{max} - y_{min}$.

After the introduction of this transformation, the convolutory integral (9) goes into the form [7,8]

$$\Delta\mu_B \cong \int_0^1 \Delta\mu(y) \cdot \frac{dr_{A,CB}(\mu, \eta)}{d\eta} \cdot d\eta \tag{11}$$

Convolutory integrals, which are given by the relations (3), (4), (9) and (11), can be also understood as the calculation of the weighted average $\Delta\mu_B$ of the quantities $\Delta\mu(.)$. The weight function is always the first derivative of the relevant WFFs and that is simultaneously the corresponding sensitivity function – the normed Green's function – see the relation (1).

A special case is the calculation of the arithmetic average. In this case, the WFFs have a specific form $r = 1 - (t_P/t_{PI})$, resp. $r = 1 - (x/x_{PI})$, resp. $r = \eta = (y - y_{min})/(y_{max} - y_{min})$.

1.4.2. Normalization of Green's functions. Norm effect

For simplicity, we will only deal with the normalization for WFFs dependent on the vertical coordinate respectively y and η, and for the notation introduced by Bliss (generalized Bliss' WFFs).

The normalization is based on the relationship for the full perturbed trajectory [4,7]

$$Q_{CP}(\mu, t_P) = \sigma_Q \cdot N_Q \cdot r(\eta) \quad \text{and} \quad (t_P = 0 \text{ and } \eta = 1) \tag{12}$$

In the case of the traditional normalization, the choose $r(1) = 1$ (as a definition) is prevalent and then

$$N_{Q1} = N_Q = \text{abs}(Q_{CP}(\mu, 0)), \tag{13}$$

$$\sigma_Q = \text{sign}(Q_{CP}(\mu, 0)), \tag{14}$$

while it is traditionally tacitly presumed that it holds $dr(\mu, y)/dy \geq 0$, then $r(\mu, y) \in \langle 0, 1 \rangle$.

If this tacit assumption $dr(\mu, y)/dy \geq 0$ is not fulfiled, i.e. that for a certain range of values of y it holds $dr(\mu, y)/dy < 0$, then also the implication $r(\mu, y) \in \langle 0, 1 \rangle$ does not hold. It can hold that $\min(r(\mu, y)) < 0$ and/or/simultaneously $\max(r(\mu, y)) > 1$, i.e. $r(1) < \max(r(\mu, y))$. The described state is called the "*norm* effect" [4,7,25].

An exact norm effect, however, occurs in the case that the $Q_P(\mu, 0) = 0$. Just then, the traditional normalization can not be used at all.

It is quite common for WFFs for virtual temperature $r_{\tau/\rho}$, which expresses only the influence of the "elasticity" of the air, i.e. the effect of the virtual temperature τ on the size of the drag coefficient $c_D(M)$ through the Mach number, resp. the speed of sound a [4,7,25]. The method of normalization for this case, which is proposed in Ref. [25], is mathematically correct, but for practical use inappropriate.

We have proposed therefore the following norm [4,7], see the equation (7)

$$N_{Q2} = N_Q = \max_y Q_{CG}(\mu, y) - \min_y Q_{CG}(\mu, y), \tag{15}$$

and at the same time

$$\sigma_Q = \text{sign}(Q_{CP}(0)), \tag{16}$$

if it applies that $Q_{CP}(t_P) \neq 0$ for $t_P = 0$, i.e. $Q_{CP}(0)$.

If $Q_{CP}(0) = 0$ then a number of varieties how to choose σ_Q exist. For example we can choose [4,7]

$$\sigma_Q = \mathrm{sign}\left(Q_{CG}\left(\mu, y_{sup}\right)\right), \tag{17}$$

where

$$\left|Q_{CG}\left(\mu, y_{sup}\right)\right| = \max_y(|Q_{CG}(\mu, y)|)$$

In case that expression $r(1) = \max(r(\mu, y))$ and $\min(r(\mu, y)) = 0$ holds, the relations (13) and (14) consistent with the traditional relation (12).

Illustrative figures to the exact norm effect are presented in Refs. [4,7].

If it is not the exact norm effect ($Q_P(\mu, 0) \neq 0$), the both norms given by relations (13) and (15) can be used. Then it is possible proceed in two steps [8]

In the first step, we use the traditional normalization according to equation (13), i.e. $r(1) = 1$. Subsequently, we calculate the course of WFFs $r(\mu, \eta)$ and we determine the norm

$$N_r = max_\eta r(\eta) - min_\eta r(\eta) = N_{Q2}/N_{Q1}, \tag{18}$$

which is analogous to the norm according to the relation (15).

In the second step, we calculate the new weighting function

$$r_N(\mu, \ \eta) = r(\mu, \eta)/N_r \tag{19}$$

for which it holds $r_N(1) = 1/N_r$ instead of the original $r(1) = 1$. In this case, it holds instead of relation (12)

$$Q_{CP}(\mu, t_P) = \sigma_Q \cdot N_{Q2} \cdot r_N(\eta) \quad \text{and} \quad (t_P = 0 \ \text{and} \ \eta = 1) \tag{20}$$

and the norm of this new WFFs $r_N(\mu, \eta)$ is equal to one ($N_r = 1$).

If the norm $N_r < \varepsilon$ and $\varepsilon <$ cc. (5–10), then the traditional norm is better for use(no or weak norm effect). In the opposite case, the norm N_{Q1} relatively small and in the traditional normalization we divide a number close to zero (strong and exact norm effect).

The following division of the WFFs can be made ($r(1) = r_N(1) \in \langle 0, 1\rangle$)

- WFFs without the norm effect ($r(1) = 1$),
- WFFs with a weak norm effect ($r(1) >$ cc. 0.2),
- WFFs with a strong norm effect ($r(1) <$ cc. 0.2),
- WFFs with the exact norm effect ($r(1) = 0$).

The problem is illustrated by the help of Figs. 1 and 2.

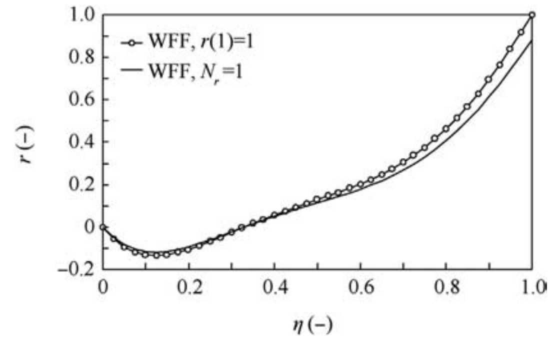

Fig. 1. Illustrative example of a WFF with the weak norm effect ($\min(r(\eta)) = -0.134\,26$ and $\max(r(\eta)) = 1.0$, $N_r = 1.134\,26$, $r_N(1) = 0.881\,6$) [8].

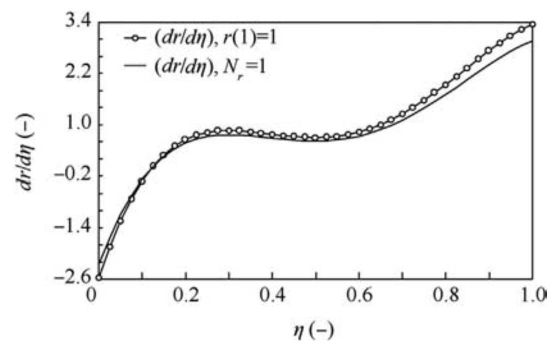

Fig. 2. Illustrative example of the normed Green's functions derived from WFFs with the weak norm effect (see Fig. 1) [8].

2. Generalized reference Height

The traditional reference height RHT is defined only semi-empirically in the textbook [10]. In this section, we derive an exact definition of the generalized reference height gRHT [8]. The traditional reference height is a special case of the generalized reference height gRHT.

2.1. Approximation of measured meteo-ballistic data

2.1.1. Basic information

We approximate a measured course of $\Delta\mu(\eta)$ by a polynomial of n-th degree using e.g. the least squares method

$$\Delta\mu(\eta) \cong \sum_{i=0}^{n} a_i \cdot \eta^i, \tag{21}$$

where

n is the degree of the polynomial. The chosen degree depends on the desired accuracy of the approximation.
a_i are the approximation constants – coefficients of the approximating polynomial.

In fact, it is necessary to proceed somewhat more complicated. The measured data $\Delta\mu(h)$ are determined in dependence on the vertical coordinate (superelevation) y_z, wherein

$$y_z = h - h_{MDP} \tag{22}$$

where

h is the altitude in which the $\Delta\mu(h)$ is measured,
h_{MDP} is the altitude of the meteo station (Meteorological Datum Plane).

The measurement of $\Delta\mu(y_z)$ takes place to the superelevation y_{zm}, i.e. $y_z \in \langle 0, y_{zm}\rangle$. The maximum superelevation value is usually 30 km ($y_{zm} \leq 30\,000$ m). It is normally $y_{zm} = 15\,000$ m [10,28,29].

We will do only one approximation of the course of $\Delta\mu(\eta_z)$

$$\Delta\mu_z(\eta_z) \cong \sum_{i=0}^{n} a_{zi} \cdot \eta_z^i, \tag{23}$$

where

$\eta_z = y_z/y_{zm}$, $\eta_z \in \langle 0, 1\rangle$

Such approximation can be easily obtained from the data given in the meteo message METCMQ [28,36] from which we can calculate the corresponding courses of $\Delta\mu_z(\eta_z)$ that can be subsequently approximated by the formula (23).

We determine the coefficients a_i by a conversion from the coefficients a_{zi}. For this, we use the link between the coordinates (relations (10 and 22))

$$y = y_z - h_{ZG}, \tag{24}$$

where$h_{ZG} = h_G - h_{MDP}$ is the gun superelevation above MDP.

To find the approximation (23), it is convenient to use procedures based on the use of orthogonal functions. The shifted Legendre polynomials (defined on the interval ⟨0, 1⟩) appear as the most advantageous. The Chebyshev or Laguerre polynomials can be also used. For the analysis of the wind can be useful to use the Fourier series, too. These problems are not the subject of this article.

2.1.2. Average meteo-data

In the meteo message METEO-11 ("Meteo-average") there is indicated the arithmetic average of the absolute and relative deviations of the meteo data [10,14–20].

In general, it holds for the average absolute deviation $\Delta\mu_{AV}(y)$ of the meteo parameter μ on the interval $\langle y_{\min}, y\rangle$ [7,8]

$$\Delta\mu_{AV}(y) = \frac{1}{y - y_{\min}} \int_{y_{\min}}^{y} \Delta\mu(y)\mathrm{d}y = \frac{1}{\Delta h}\int_0^{\Delta h} \Delta\mu(\Delta h)\mathrm{d}\Delta h = \frac{1}{\eta}\int_0^{\eta} \Delta\mu(\eta)\mathrm{d}\eta \tag{25}$$

An analogous relationship is true also for the *average relative deviation* $\delta\mu_{AV}(y)$ of the meteo parameter μ on the interval $\langle y_{\min}, y\rangle$, while $\delta\mu_{AV}(y) = \delta\mu_{AV}(h) = \Delta\mu(h)/\mu_{STD}(h)$. The relative deviations for the wind are not defined.

After substituting relation (21) into (25) and the adjustment, we obtain the approximate relationship for the average absolute deviation $\Delta\mu_{AV}(y)$ of the meteo parameter μ on the interval $\langle y_{\min}, y\rangle$ [7,8]

$$\Delta\mu_{AV}(\eta) \cong \sum_{i=0}^{n} \left(\frac{a_i}{(i+1)}\right)\cdot\eta^i = \sum_{i=0}^{n} b_i\cdot\eta^i, \tag{26}$$

where

$b_i = a_i/(i + 1)$ are the new approximation constants – coefficients of the approximating polynomial. It is clear from the relation between a_i and b_i that averaging is a special case of the low frequency filtration of the input/output signal $\Delta\mu(y)$.

In the case of the absolute deviations given in the METEO-11 (meteo-average), the analogous approximation to (26) derived from (23) is valid

$$\Delta\mu_{AV,z}(\eta_z) \cong \sum_{i=0}^{n} \left(\frac{a_{zi}}{(i+1)}\right)\cdot\eta_z^i = \sum_{i=0}^{n} b_{zi}\cdot\eta_z^i, \tag{27}$$

where

$b_{zi} = a_{zi}/(i + 1)$ are the new approximation constants – coefficients of the approximation polynomial. It is again obvious from the relationship between a_{zi} and b_{zi} that the averaging is a special case of low-frequency filtration of the input/ouput signal $\Delta\mu(y)$.

It is therefore obvious that we can proceed the other way around. Firstly we approximate the pre-calculated average absolute deviation $\Delta\mu_{AV}(y)$ of the meteo parameter μ on the interval $\langle y_{\min}, y\rangle$, to which we can use, for example, the data referred to in the METEO-11 ("Meteo-average"). So primarily we obtain the coefficients b_{zi} and b_i and only subsequently we can calculate the size of the coefficients a_{zi} and a_i.

2.2. Moments of weighting factor functions

If we substitute into relation (11) the expression (21), then we receive after integration and adjustment the working relationship for the *absolute ballistic deviation*/perturbation $\Delta\mu_B$ of the meteo parameter μ [7]

$$\Delta\mu_B \cong \sum_{i=0}^{n} a_i\cdot m_{WFF,i}, \tag{28}$$

where

$$m_{WFF,i} = \int_0^1 \eta^i\cdot\frac{dr(\mu,\eta)}{d\eta}\cdot d\eta,$$

$m_{WFF,i}$, $i = 0, 1, 2, \ldots, n$, are the normed moments of the first derivative of the weighting function WFF $r(\mu,\eta)$, so the normed Green's function. They are therefore the characteristics of the sensitivity function.

For the first two normed moments apply

$$m_{WFF,0} = r(\mu,1) = r(1), \tag{29}$$

and

$$m_{WFF,1} = \int_0^1 \eta\cdot\frac{dr(\mu,\eta)}{d\eta}\cdot d\eta \tag{30}$$

The use of normed moments of the first derivative of the weighting function WFF has not yet been published in the available literature.

If we substitute into relation (11) the expression (26), then we receive after integration and adjustment a working relationship for the *absolute ballistic deviation*/perturbation $\Delta\mu_B$ of the meteo parameter μ [7]

$$\Delta\mu_B \cong \sum_{i=0}^{n} b_i\cdot m_{AV,i}, \tag{31}$$

where

$$m_{AV,i} = (i+1)\cdot m_{WFF,i},\ i = 0, 1, \ldots, n, \tag{32}$$

and so for $i = 0$ and 1

$$m_{AV,0} = m_{WFF,0} = r(\mu,1) = r(1), \tag{33}$$

$$m_{AV,1} = 2\cdot m_{WFF,1}. \tag{34}$$

Most of the weighting functions WFF $r(\mu, \eta)$ is determined numerically, so they are known only their discrete values $r_j = r(\mu, \eta_j)$, $j = 0, 1, \ldots$. If we use for the numerical integration of the relation (28) the rectangular rule, we obtain the following working relations for the determination of the normed moments of the first derivative of the weighting function WFF

$$m_{WFF,i} = \sum_{j=1}^{m} \left(\frac{\eta_j + \eta_{j-1}}{2}\right)^i\cdot q_j, \tag{35}$$

where

$q_j = r_j - r_{j-1} = r(\mu, \eta_j) - r(\mu, \eta_{j-1})$ are the discrete weights/weight numbers (weighting factors) WFs for the j-th layer/zone [1,6,8,10,19,22,24–29],

$\Delta\eta_j = \eta_j - \eta_{j-1}$ relative height of the j-th layer/zone.

For $j = 0$ is $\eta_{j=0} = \eta_0 = 0$ and $r_{j=0} = r_0 = 0$.
For $j = m$ is $\eta_{j=m} = \eta_m = 1$ and $r_{j=m} = r_m = r(\mu, 1) = r(1)$.

For example, it is true for the weight function WFF, derived under assumption, that the projectile trajectory is approximately equal to the projectile's trajectory in a vacuum [1,6–8,10,24,26]

$$r_V(\eta) = 1 - \sqrt{1-\eta}. \tag{36}$$

The normed moments of the first derivative of the weighting function WFF $m_{WFF,j}$, $j = 0, 1, \ldots, 5$ for this relationship, calculated using the formula (35) with the step $\Delta\eta = 0.002$, are 1; 0.6666; 0.5333; 0.4571; 0.4063; 0.3693.

For the WFF ($r(1) = 1$) in the Fig. 1, it holds accordingly 1; 0.7970; 0.6334; 0.5303; 0.4585; 0.4048.

2.3. Generalized reference height of projectile trajectory

We use the linear approximation [8] for the *absolute ballistic deviation/perturbation* $\Delta\mu_B$ of the meteo parameter μ under utilization of the relations (31, 33, 34)

$$\Delta\mu_B \approx b_0 \cdot r(1) + b_1 \cdot m_{AV,1}. \tag{37}$$

We make the following hypothesis: For the generalized reference height Y_{CR}, respectively, for the corresponding η_{CR} must be true

$$\Delta\mu_B = b_0 + b_1 \cdot \eta_{CR} \tag{38}$$

By comparing the relations (37 and 38), we obtain the relationship for the coefficient of the generalized reference height [8].

$$K_{CR} = \eta_{CR} \approx m_{AV,1} - (1 - r(1)) \cdot \left(\frac{b_0}{b_1}\right) = 2 \cdot \left[m_{WFF,1} - (1 - r(1)) \cdot \left(\frac{a_0}{a_1}\right)\right], \tag{39}$$

and it holds for the generalized reference height [8].

$$Y_{CR} = \Delta h_{CR} = K_{CR} \cdot \Delta h_m \tag{40}$$

Relations (39,40) represent an exact definition of the generalized reference height Y_{CR} and are the result of our research. They have not yet been published. This is a characteristic of the sensitivity function – the normed Green's function.

In the case that $y_{min} = 0$, then $\Delta h_m = y_{max}$ and the relationship (40) defines the reference height Y_R in the traditional concept [1,10].

The second member of relation (39) will be zero in two cases:

- if it will be used the traditional standardization (relations (13,14), $r(1) = 1$) – this requirement can be satisfy, if respectively no strong and no exact norm effect is generating – the paragraph 1.4.2.
- if it will be true specially, that $a_0 = 0$. This can be achieved, if we use the regression function (relation (21)) in the special form $\Delta\mu(\eta) = a_1 \cdot \eta$.

If $a_1 = 0$, then immediately $\Delta\mu_B \approx a_0$.

As we stated in the paragraph 2.1.1, the measured meteo data are recorded in dependence on the vertical coordinate y_z in the meteo messages, then it is preferable to find a relationship for the generalized reference height $Y_{CR,z}$, transformed to the coordinates y_z [7,8,10]. We derive the transformation relationship now. The approximate transformation relationship now we derive with the use of Fig. 3.

First we find the transformation relations between the (a_{z0}, a_{z1}) and (a_0, a_1) – relations (21,23 and 10) assuming that the approximations are linear. Substituting for y_z corresponding values of y given by the equation (24) into the linearized equation (23) we obtain the two approximations

$$\Delta\mu_{AV,z}(y) \cong a_{z0} + a_{z1} \cdot \left(\frac{y + h_{ZG}}{y_{zm}}\right), \tag{41}$$

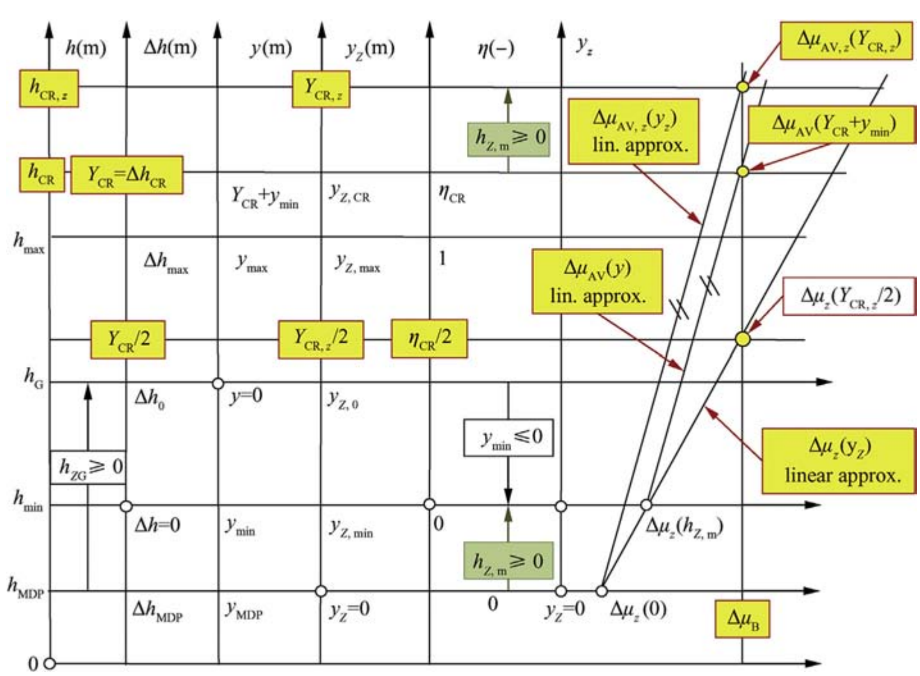

Fig. 3. The relationships between the different coordinate systems and linearized courses of the quantities $\Delta\mu_z(y_z)$ – the relation (23), $\Delta\mu_{AV,z}(y_z)$ (data from the METEO-11/meteo-average) – the relation (27) and $\Delta\mu_{AV}(y)$ – the relation (26).

$$\Delta\mu_{AV}(y) \cong a_0 + a_1 \cdot \left(\frac{y - y_{min}}{\Delta h_m}\right) \quad (42)$$

Because this is about the same altitude, or the ordinate y, both expressions must equal each other. From the equality of the expressions and by comparing term by term we obtain the transformation relations

$$a_1 = (\Delta h_m / y_{zm}) \cdot a_{z1}, \quad (43)$$

$$a_0 = a_{z0} + a_{z1} \cdot (\Delta h_{Z,m} / y_{zm}), \quad (44)$$

where

$\Delta h_{Z,m} = h_{ZG} + y_{\min}$ is the linear shift of the origin of the coordinate system y_z and Δh, further $y_{\min} \le 0$ [4,7,8].

We make the hypothesis that it approximately applies

$$\Delta\mu_B \approx b_{z0} + b_{z1} \cdot \left(\frac{Y_{CR,z}}{y_{zm}}\right) \quad (45)$$

Now we compare the relations (38 and 45) using (40) and simultaneously we account the relations between the (a_{z0}, b_{z0}) and (a_0, b_0), (a_{z1}, b_{z1}) and (a_1, b_1) given by (26 and 27).

We receive a wanted approximate transformation relationship

$$Y_{CR,z} \approx Y_{CR} + \Delta Y_{CR,z}, \quad (46)$$

where

$\Delta Y_{CR,z} = 2 \cdot \Delta h_{Z,m} = 2 \cdot (h_{ZG} + y_{\min})$ is the repair of the generalized reference height, $y_{\min} \le 0$. This is a generalization of the relation cited in the literature [10,14,16,18] for the case $y_{\min} = 0$.

Search for the absolute ballistic deviation/perturbation $\Delta\mu_B$, in the meteo message METEO-11 (meteo-average), is performed in two approximate ways.

In the first case, it approximately applies

$$\Delta\mu_B \approx \Delta\mu_{AV,z}(y_z) \text{ and } y_z = Y_{CR,z} \quad 47)$$

In the second case, the first performs a transformation of all the coordinates y_z in the meteo message

$$y_{z,red} = y_z - \Delta Y_{CR,z} \quad (48)$$

and subsequently approximately applies

$$\Delta\mu_B \approx \Delta\mu_{AV,z}(y_{z,red}) \text{ and } y_{z,red} = Y_{CR} \quad (49)$$

In both cases, the calculation of the regression coefficients $b_{z0} = a_{z0}$ and $b_{z1} = a_{z1}/2$ (the relation (23)) at all does not perform.

According to the traditional recommendations ($y_{\min} = 0$) is taken, that $\Delta Y_{CR,z} = 0$, if abs(h_{ZG}) $\le$ 200 m [7,10,14,16,18].

It follows from the Fig. 3 that an approximation of $\Delta\mu_z(y_z)$ can be also used. We obtain this approximation from the data given in the meteo message METCMQ [28,36] (and also see the comment to the relation (23)), while the estimate of the value of $\Delta\mu_B$ we calculate for $y_z = Y_{CR,z}/2$.

3. Alternative relations for the determination of ballistic deviations

The aim is to find convolutory integral analogous to the relation (11) in which would perform the average absolute deviation $\Delta\mu_{AV}(\eta)$ instead of the measured absolute deviation $\Delta\mu(\eta)$.

3.1. Alternative analytic relation

In the derivation we use the two basic relations [7].

The first relation is a differential equation for the $\Delta\mu_{AV}(\eta)$ obtained by the differentiation of the relation (25)

$$\eta \cdot \frac{d\Delta\mu_{AV}(\eta)}{d\eta} = \Delta\mu(\eta) - \Delta\mu_{AV}(\eta) \quad (50)$$

The second relation is a well-chosen expression [7].

$$\frac{d}{d\eta}\left[\eta \cdot \Delta\mu_{AV}(\eta) \cdot \frac{dr}{d\eta}\right] \quad (51)$$

We derivative this expression so that we could use the method of integration by parts. To the resulting expression we substitute relation (50) and after adjustment we obtain [7].

$$\Delta\mu(\eta) \cdot \frac{dr}{d\eta} = \frac{d}{d\eta}\left[\eta \cdot \Delta\mu_{AV}(\eta) \cdot \frac{dr}{d\eta}\right] - \eta \cdot \Delta\mu_{AV}(\eta) \cdot \frac{d^2 r}{d\eta^2} \quad (52)$$

We substitute the expression (52) into the relation (11) and perform the integration, so we obtain a new relation for the absolute ballistic deviation/perturbation $\Delta\mu_B$ of the meteo parameter μ [7] in dependence on the $\Delta\mu_{AV}(\eta)$

$$\Delta\mu_B = \left[\eta \cdot \Delta\mu_{AV}(\eta) \cdot \frac{dr(\eta)}{d\eta}\right]_{\eta=1} - \int_0^1 \Delta\mu_{AV}(\eta) \cdot \left[\eta \cdot \frac{d^2 r}{d\eta^2}\right] \cdot d\eta, \quad (53)$$

This formula is not cited in the literature. The relationship is the result of our research. Its use is strongly restricted by the necessity to carry out for the majority of the WFFs limit transitions in the derivation of working relations.

3.2. Numerical approximation formula

The formula (53), although not in the literature cited, yet its numerical version is given in Ref. [10]. The following procedure [7] corresponds with the procedure referred to in Ref. [10].

We go out of the relationship (11) and the expressions for the weighting factors q – the relation (35), then

$$\begin{aligned}\Delta\mu_B &\cong \int_0^{\Delta h_m} \Delta\mu(\Delta h) \cdot \frac{dr(\mu, \Delta h)}{d\Delta h} \cdot d\Delta h \\ &\cong \sum_{j=1}^{m} \int_{\Delta h_{j-1}}^{\Delta h_j} \Delta\mu(\Delta h) \cdot \left(\frac{q_j}{\Delta\Delta h_j}\right) \cdot d\Delta h \\ &= \sum_{j=1}^{m} \left(\frac{q_j}{\Delta\Delta h_j}\right) \cdot \int_{\Delta h_{j-1}}^{\Delta h_j} \Delta\mu(\Delta h) \cdot d\Delta h,\end{aligned} \quad (54)$$

The integral in this relation can be with the use of the relation (25) expressed in the form

$$\int_{\Delta h_{j-1}}^{\Delta h_j} \Delta\mu(\Delta h) \cdot d\Delta h = \Delta h_j \cdot \Delta\mu_{AV}(\Delta h_j) - \Delta h_{j-1} \cdot \Delta\mu_{AV}(\Delta h_{j-1}), \quad (55)$$

We substitute the expression (55) into (54)

$$\Delta\mu_B \cong \sum_{j=1}^{m}\left(\frac{q_j}{\Delta\Delta h_j}\right)\cdot\left[\Delta h_j\cdot\Delta\mu_{AV}(\Delta h_j) - \Delta h_{j-1}\cdot\Delta\mu_{AV}(\Delta h_{j-1})\right], \quad (56)$$

Now we do rearrange the members in the sum, so as to create terms with the same Δh_j and finally we pass from the variable Δh to the variable η. So we get the following generalized expression for the numerical estimate of the absolute ballistic deviation/perturbation $\Delta\mu_B$ of the meteo parameter μ [7]

$$\Delta\mu_B \cong \sum_{j=1}^{m-1}\Delta\mu_{AV}(\eta_j)\cdot K_j + \Delta\mu_{AV}(\eta_m = 1)\cdot K_m, \quad (57)$$

where

$$K_j = \left[\frac{q_j}{\Delta\eta_j} - \frac{q_{j-1}}{\Delta\eta_{j-1}}\right]\cdot\eta_j, \quad j = 1, 2, \ldots, (m-1), \quad (58)$$

$$K_m = \frac{q_m}{\Delta\eta_m} \quad (59)$$

The formula (57) is a numeric version of the relation (53).

3.3. The traditional relationship for the reference height

We obtain the relations for the reference height Y_R also by adjusting the relationship (57) [7,10]. We generalize the procedure that we used in the article [1] and the related study [6].

The basic simplification lies in the fact that the WFF is approximated only by using the two weighting factors WFs q_{12}, q_{22} for intervals $\langle 0, \Delta h_{RR}\rangle$ and $\langle \Delta h_{RR}, \Delta h_m\rangle$, i.e. $m = 2$ in the relation (57). Two values $\Delta\mu_{AV}(\Delta h_{RR})$ and $\Delta\mu_{AV}(\Delta h_m)$ follow from the relation (25). At the same time the *only two options* ($q_{12} = 0$ and $q_{22} = 1$) or ($q_{12} = 1$ and $q_{22} = 0$) hold good. The aim is to find the appropriate approximation relations for the height Δh_{RR}.

We introduce the estimates of the partial derivatives

$$r'(\Delta h) = \frac{\partial r}{\partial \Delta h} \cong r'_{12} = \frac{q_{12}}{\Delta h_{RR}} \quad (60)$$

$$r'(\Delta h) = \frac{\partial r}{\partial \Delta h} \cong r'_{22} = \frac{q_{22}}{\Delta h_m - \Delta h_{RR}}, \quad (61)$$

and put them into relation (54)

$$\Delta\mu_B = \int_0^{\Delta h_m}\Delta\mu(\Delta h)\cdot r'_A(\Delta h)\cdot d\Delta h \cong r'_{12}\cdot\int_0^{\Delta h_{RR}}\Delta\mu(\Delta h)\cdot d\Delta h + r'_{22}\cdot\int_{\Delta h_{RR}}^{\Delta h_m}\Delta\mu(\Delta h)\cdot d\Delta h + r'_{22}\cdot\left[\int_0^{\Delta h_{RR}}\Delta\mu(\Delta h)\cdot d\Delta h - \int_0^{\Delta h_{RR}}\Delta\mu(\Delta h)\cdot d\Delta h\right]$$

After the adjustment we receive the *basic calculation relationship* for the *numerical estimate* for the *absolute ballistic deviation/* perturbation $\Delta\mu_B$ of the meteo parameter μ (($q_{12} = 0$ and $q_{22} = 1$) or ($q_{12} = 1$ and $q_{22} = 0$))

$$\Delta\mu_B \cong q_{12}\cdot\Delta\mu_{AV}(\Delta h_{RR}) + q_{22}\cdot\Delta\mu_{AV}(\Delta h_{RR}, \Delta h_m), \quad (62)$$

where

$$\Delta\mu_{AV}(\Delta h_{RR}, \Delta h_m) \cong \frac{1}{\Delta h_m - \Delta h_{RR}}\left[\Delta h_m\cdot\Delta\mu_{AV}(\Delta h_m) - \Delta h_{RR}\cdot\Delta\mu_{AV}(\Delta h_{RR})\right], \quad (63)$$

it is the average $\Delta\mu_{AV}(\Delta h)$ in the interval $\langle\Delta h_{RR}, \Delta h_m\rangle$.

Finally, we determine the area under the WFF

$$S = \int_0^1 r(\eta)\cdot d\eta \quad (64)$$

Further analysis leads to the derivation of the two values Δh_{RR} and Δh_{R2} for the concave shape ($q_{12} = 1$, $q_{22} = 0$, $S \geq 0.5$; $\Delta h_{RR} = \Delta h_{R2}$) of the weighting function WFF – Fig. 4 and Δh_{R1} for the convex shape ($q_{12} = 0$, $q_{22} = 1$, $S < 0.5$; $\Delta h_{RR} = \Delta h_{R1}$) of the weighting function WFF – Fig. 5. Finally, it is derived another relation for the *generalized reference height* [7]

$$Y_{CR} = \Delta h_{R2} = \Delta h_{R1} + \Delta h_m = K_R\cdot\Delta h_m, \quad (65)$$

where

$$K_{CR} = K_R \cong 2\cdot(1 - S) \quad (66)$$

The relationship (65) is a generalization of the relations derived in Refs. [1,6,10].

A deeper analysis of given problems can be found in Refs. [1,6,10]. So far, we did not check the accuracy of the relationship for the case of the existence of a norm effect – see the paragraph 1.4.2.

4. Generalized Petrovic algorithm for the reference height calculation

In the article [20], Petrovic suggested an effective algorithm for the calculation of the rereference height. It is not necessary to count the whole course of the WFF or its first derivative, but it is sufficient to calculate the projectile trajectory only three times. He deduced the competent relationship by an intuitive reasoning. In this section, we derive the competent relationship correctly and – in addition – we generalize Petrovic algorithm for the calculation of the moments of the first derivative of the WFF.

4.1. Generalized Petrovic algorithm

We will not proceed completely in general, but we will consider only the perturbation of the range $\Delta X(\mu)$ at a constant height of impacts of the projectiles ($\Delta Y(\mu) = 0$, the iso-height perturbations [4,7]).

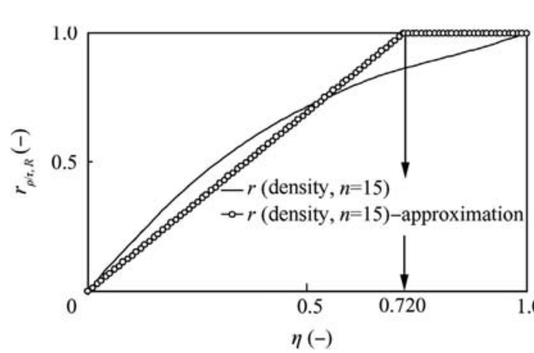

Fig. 4. WFF for the air density ρ (concave shape) $r_{\rho/\tau,R}(\eta)$, Y_s = 18 000 m, S = 0.640, K_{CR} = 0.720, $Y_R = \Delta h_{R2}$ = 13 000 m [1,6].

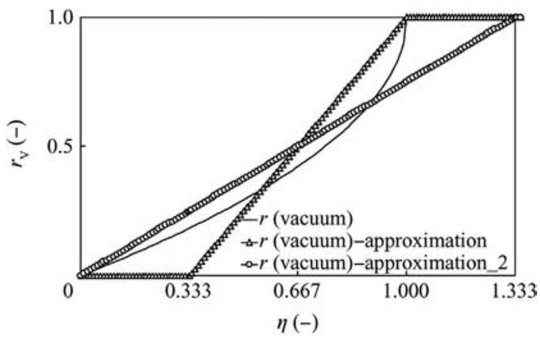

Fig. 5. WFF – vacuum (convex shape) $r_V(\eta)$, relation (36), $S = 0.333$, $K_{CR} = 1.333$ [1,6].

To calculate the reference height Y_{CR}, it is sufficient to calculate the projectile trajectory only three times. The first track is standard (unperturbed, paragraph 1.4.) calculated for $\mu(\eta) = \mu_{STD}(\eta)$. The next two tracks are fully perturbed and are calculated for the following course $\mu(\eta)$

$$\mu_j(\eta) = \mu_{STD}(\eta) + \Delta\mu_j(\eta) \quad ,j = 0, 1, \tag{67}$$

$$\Delta\mu_j(\eta) = \Delta\mu_0 \cdot \varphi_j(\eta) = \Delta\mu_0 \cdot [a_{0j} + a_{1j} \cdot \eta], \tag{68}$$

where

$\Delta\mu_0 = \Delta z_{m0} =$ const the excitation amplitude – the relation (1), more closely [4],

$\varphi_j(\eta)$ are two special choices of approximation according to relation (21), $j = 0, 1$.

By calculations we get the ranges – the standard X_{STD} and two fully perturbed X_0 and X_1. Models with 3, 4/5 or 6 degrees of freedom (DOF) can be used to the calculation. Subsequently, we calculate the (full) perturbation of the trajectory

$$\Delta X_j = X_j - X_{STD}, \quad j = 0, 1 \tag{69}$$

In accordance with the relations (2,3 and 27), it is true ($\Delta\mu_{B0} = 1$)

$$\Delta X_j = \Delta X_N \cdot \vartheta_j, \tag{70}$$

where

$\Delta X_N = \sigma_Q \cdot N_Q \cdot \Delta\mu_0$ is the norming value and

$$\vartheta_j = a_{0j} \cdot m_{WFF,0} + a_{1j} \cdot m_{WFF,1} = a_{0j} \cdot r(1) + a_{1j} \cdot m_{WFF,1}, \quad j = 0, 1 \tag{71}$$

The relations (69–71) constitute a system of two equations in one parameter $m_{WFF,0} = r(1)$ and the one unknown $m_{WFF,1}$. We search the solution provided that $a_{10} = 0$, then

$$K_{CR,1} = \frac{m_{WFF,1}}{r(1)} = \frac{1}{a_{11}}\left[a_{00} \cdot \left(\frac{\Delta X_1}{\Delta X_0}\right) - a_{01}\right] \tag{72}$$

This relationship is a generalization of the relation (77), which was derived by Petrovic with an intuitive technique. The relationship is only valid provided that $a_{11} \neq 0$ and an exact norm effect does not occur ($r(1) = 0$ and $\Delta X_0 = 0$). We do not recommend to use it, if a strong norm effect arises ($r(1) \to 0$). Petrovic probably didn't know anything about the norm effect and unwittingly assumed that $r(1) = 1$ and $\Delta X_0 \neq 0$ [20].

We express $m_{WFF,1} = r(1) \cdot K_{CR,1}$ from the relation (72) and substitute it into the relation (39), then

$$K_{CR} = 2 \cdot \left[r(1) \cdot K_{CR,1} - (1 - r(1)) \cdot \left(\frac{a_{01}}{a_{11}}\right)\right] \tag{73}$$

If the traditional standardisation will be used ($r(1) = 1$), it will be

$$K_{CR} = 2 \cdot K_{CR,1} \tag{74}$$

In accordance with the Petrovic we choose

$$a_{00} = 1, \ a_{01} = -1, \ a_{11} = 2, \tag{75}$$

then the relation (72) goes over into the shape

$$K_{CR,1} = \frac{m_{WFF,1}}{r(1)} = \frac{1}{2}\left[1 + \left(\frac{\Delta X_1}{\Delta X_0}\right)\right] \tag{76}$$

If $r(1) = 1$, then the relation (74) will have a special shape

$$K_{CR} = 1 + \left(\frac{\Delta X_1}{\Delta X_0}\right) \tag{77}$$

Petrovic derived this relationship by an intuitive procedure. Petrovic further assumed traditionally in the relation (40) that $\Delta h_m = Y_S$.

Our exact derivation shows that the relation (77) is only a special case of the relation (73), which respects the existence of the norm effect.

If we know the generalized reference height Y_{CR} (relation (40)) and the height Δh_m, then the K_{CR} can be calculated. If we know moreover $r(1)$ and the coefficients a_{ij}, $i = 0, 1$; $j = 0, 1$, e.g. according to (75), and if we calculated or acquired the value ΔX_0 from the Tabular Firing Tables, then the size ΔX_1 can be calculated easily from the relations (72,73).

4.2. Effective algorithm for the calculation of the WFF moments

We don't know, if Petrovic knew, that the coefficients (75), which he used, are the coefficients of the first two shifted Legendre polynomials. It inspired us to a generalization of the algorithm described in the previous paragraph.

If we use the first $(n + 1)$ shifted Legendre polynomials as the function $\varphi_j(\eta)$, $j = 0, 1, \ldots, n$ in relations (67 and 68), and we calculate $(n + 2)$ projectiles trajectories, from which one standard and $(n + 1)$ perturbed (calculation of ΔX_j, $j = 0, 1, \ldots, n$), then it can be calculated by indicated way the n moments $m_{WFF,j}$, $j = 1, 2, \ldots, n$, if $r(1) \neq 0$.

In the case that $r(1) = 0$ (the exact norm effect), a parametric solution can be found and if we know one nonzero moment $m_{WFF,k}$, $k \neq 0$, then we can calculate the remaining $(n - 1)$ moments.

Implied procedure [8] is numerically very economical and has not yet been published anywhere. It is about the generalization of the system of two linear equations (69) and (70) to the system of $(n + 1)$ equations

$$\vartheta_j = \sum_{i=0}^{j} a_{ij} \cdot m_{WFF,j}, \quad j = 0, 1, 2, \ldots, n, \tag{78}$$

where

a_{ij} are the coefficients of the first $(n + 1)$ shifted Legendre polynomials. The coefficients form the matrix of the system $\boldsymbol{A}$, that it is a lower triangular, so the solving the system (78) is algorithmically undemanding.

5. Conclusion

In this contribution and in the article [4], we have presented the core of the improved theory of generalized WFFs and their moment characteristics, specifically generalized reference height of trajectory, as a special sensitivity functions.

This theory allows to perform an effective sensitivity analysis of the properties of non-standard projectile trajectories. The theory is fully linked to the more general theory of sensitivity analysis of dynamical systems, so the results can be interpreted in a broader context.

The theory of generalized reference height Y_{CR} published in this contribution creates the potential for a major simplification and an improvement in the calculations of corrections on the changes of the meteo ballistic parameters μ compared to the traditional method [10,14–18,22–29,32,33] using the tabulated values of the weighting factors q and which is used almost unchanged since the 1920s. Our creation of conditions for the extension of the use of RHT even to the perturbation model used in NATO, is important benefit.

The theory of generalized reference heights assumes that data about the meteo parameters processed according to the methodology of formation of the METEO-11 ("meteo-average") are available. The principle of this methodology was clarified in this contribution and the articles [1,4]. The necessary initial data can be simply calculated from the data given in the meteo message METCMQ [36] (wind vector $\boldsymbol{w}(y_z)$ and virtual temperature $\tau(y_z)$ for $y_z \in \langle 0, y_{zm} \rangle$, pressure $p_{0,MDP}$ at MDP level).

In the next period, we plan to publish the contributions to the problems of the numerical calculation of the WFFs, since it is associated with a number of non-trivial inconveniences.

Consequently, we will continue in the research of the evaluation of the accuracy of the published algorithms.

Acknowledgments

This work originated with the support of financing from the Research Project for the Development of the Department of Weapons and Ammunition, Faculty of Military Technology, University of Defence, Brno, DZRO VYZBROJ.

References

[1] Cech V, Jedlicka L, Jevicky J. Problem of the Reference Height of the Projectile Trajectory as a Reduced Meto-ballistic Weighting Factor. Defence Technology 10, Number 2 (June 2014), Issue 2: Special issue of the 28th International Symposium on Ballistics, ISSN: 2214 – 9147, p. 131–140.

[2] Cech V, Jedlicka L, Jevicky J. Problem of the Reference Height of the Projectile Trajectory as a Reduced Meto-ballistic Weighting Factor. Extended Abstract. Proceedings of the 28th International Symposium on Ballistics, Editors: Ames, R. G., Boeka, R. D., Atlanta, September 22-26, 2014, p. 7 – 10.

[3] Cech V, Jedlicka L, Jevicky J. Some Problems with the Estimation of Projectile Trajectory Perturbations. Proceedings of 20th International Conference Engineering Mechanics 2014, Book of full texts, Editor: Fuis, V., Svratka, May 12 – 15, 2014, p. 116 – 119.

[4] Cech V, Jevicky J. Improved theory of generalized meteo-ballistic weighting factor functions and their use, Defence Technology 12 (June 2016), Issue 3: Special issue of the 29th International Symposium on Ballistics, ISSN: 2214 – 9147, p. 242–254.

[5] Cech V, Jevicky J. Improved theory of generalized meteo-ballistic weighting factor functions and their use, Extended abstract. Proceedings of the 29th International Symposium on Ballistics, Volume 1, Editors: Woodley, C., Cullis, I., Edinburgh, May 9 -13, 2016, p. 92–95.

[6] Cech V. Perturbation model of the projectile trajectory (in Czech). Research report. Brno: University of Defence; 2014. p. 114.

[7] Cech V. Reference height of the projectile trajectory (in Czech). Research report. Brno: University of Defence; 2015. p. 128.

[8] Cech V. Meto - ballistic corrections of the projectile trajectory (in Czech). Research report. Brno: University of Defence; 2016. p. 57.

[9] STANAG 4635 The NATO Error Budget Model.

[10] Kovalenko VV, Shevkunov VI. Meteorological preparation of artillery fire (in Russian). Leningrad: Military Artillery Academy of M. I. Kalinin; 1975. p. 84.

[11] Zipfel PH. Modeling and simulation of aerospace vehicle dynamics. 2nd ed. Reston, VA, USA: AIAA, Education series; 2007. p. 607.

[12] Bockarev AF, Andrejevskij VV, Belokonov VM. Aeromechanics of the aircraft. Dynamics of flight (in Russian). Mashinostroyenie, Moscow. 1985. p. 360.

[13] Golubev IS, Svetlov VG, et al. Design of anti-aircraft missiles. 2nd ed. Moscow: MAI; 2001. p. 732.

[14] Logvin AM, Aleksandrov VI. Gunnery and exterior ballistics (in Russian). Penza: Military Artillery Academy of N. N. Voronov; 1977. p. 255.

[15] Konovalov AA, Nikolajev Ju V. Exterior ballistic (in Russian). CNII; 1979. p. 228.

[16] Composite authors. Del–55–26 fire and fire control of the field artillery (in Czech). [Textbook]. Prague: Ministry of Defence of the Czechoslovak Socialistic Republic; 1981. p. 856.

[17] Dmitrijevskij AA, Lysenko LN, Bogodistov SS. Exterior ballistic (in Russian). 3rd ed. Moscow: Mashinostroyenie; 1991. p. 640.

[18] Lysenko LN, Benevolskij SV, Burlov VV, Kazakovcev VP, et al. Ballistics. Textbook (in Russian). Penza: Artillery engineering institute of Penza; 2005. p. 512.

[19] Petrovic DR. Mechanized procedure for the calculation of altitude coefficients. In: Scientific-technical review, LV; 2005. p. 9–14. No. 3–4.

[20] Petrovic DR. New procedure for calculating altitude coefficients. In: Scientific-technical review, LVI; 2006. p. 47–51. No. 3–4.

[21] Bliss GA. Functions of lines in ballistics. In: Transactions of the American mathematical society, vol. 21; Apr. 1920. p. 93–106. No. 2.

[22] Cranz C. Textbook of ballistics. First volume. Exterior ballistics (in German). 5th ed. Berlin: Springer Verlag; 1925. p. 712.

[23] Bliss GA. Mathematics for exterior ballistics. London: John Wiley and Sons, Inc.; 1944. p. 128. Printed in USA.

[24] Curti P. Introduction into exterior ballistics (in German). Verlag Huber and Co., Frauenfeld; 1945. p. 408.

[25] McShane EJ, Kelley JL, Reno FV. Exterior ballistics. Denver: University of Denver Press; 1953. p. 834.

[26] Molitz H, Strobel R. Exterior ballistics (in German). Berlin: Springer-Verlag; 1963. p. 610.

[27] STANAG 4061 MET Adoption of a Standard Ballistic Meteorological Message (METBKQ). 2000.

[28] Composite authors. Artillery meteorology, Department of the Army 1970, FM 6 –15 C1, p. 312.

[29] Composite authors. Tactics, techniques and procedures for field artillery meteorology, Department of the Army 2007, FM 3 – 09.15, p. 270.

[30] Dickinson ER. The production of firing tables for cannon artillery. BRL report R 1371. Aberdeen proving ground, MA: RDT&E project 1T523 801A 287; 1967. p. 110.

[31] Longdon LW, et al. Textbook of ballistics and gunnery. Volume one and twovol. 1987. London: Her Majesty's Stationery Office; 1984. p. 806, 521.

[32] Composite authors. Tactics, techniques and procedures for field artillery, manual cannon gunnery. Department of the Army; 1996. p. 757. FM 6 – 40.

[33] STANAG 4119 Adoption of a Standard Cannon Artillery Firing Table Format. 2007.

[34] Rosenwasser Je N, Jusupov RM. Sensitivity of control systems (in Russian). Moscow: Nauka; 1981. p. 464. p..

[35] Kato T. Perturbation theory for linear operators. Reprint of the 1980 edition. Berlin, Heidelberg: Springer –Verlag; 1995. p. 620.

[36] STANAG 4082 MET Adoption of a Standard Artillery Computer Meteorological Message (METCMQ).